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On Weight Functions Admitting Chebyshev Quadrature

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Abstract. In this paper we prove the existence of Chebyshev quadrature for three new weight functions which are quite different from the two known examples given by Ullman [15] and Byrd and Stalla [2]. In particular, we indicate a simple method to construct weight functions for which there exist infinitely many Chebyshev quadrature rules.

1. Introduction. By a weight function w we mean a real-valued nonnegative function on $[-1, 1]$ for which the proper or improper Riemann integral exists and has positive value. We shall consider quadrature rules Q_n of the type

$$Q_n[f] := \sum_{\nu=1}^{n} a_{\nu,n} f(x_{\nu,n}),\qquad \int_{-1}^{1} f(x)w(x)\,dx = Q_n[f] + R_n[f] \tag{1}$$

having real nodes $x_{\nu,n}$ and real weights $a_{\nu,n}$.

A quadrature rule (1) is called a Chebyshev quadrature rule (in the strict sense) if the following holds:

$$a_{1,n} = a_{2,n} = \cdots = a_{n,n}, \tag{2}$$

$$-1 < x_{1,n} < x_{2,n} < \cdots < x_{n,n} < 1, \tag{3}$$

$$R_n[f] = 0 \quad \text{for all } f \in \mathcal{P}_n. \tag{4}$$

($\mathcal{P}_n$ denotes the class of polynomials of degree $\leqslant n$.) We say that a weight function w admits Chebyshev quadrature if there exist Chebyshev quadrature rules Q_n for all positive integers n.

The study of Chebyshev quadrature rules began in 1874 with the classical paper of Chebyshev [3]. Since then, there have been further investigations in the mathematical literature. For a review of recent advances in this field we refer to the paper of Gautschi [7].

Until 1966, the only known weight function admitting Chebyshev quadrature was the Chebyshev weight function

$$w_1(x) = (1 - x^2)^{-1/2}. \tag{5}$$

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In 1966 Ullman [15] proved that the weight function

$$(6) \qquad w_2(x) = w_1(x)\frac{1+ax}{1+a^2+2ax}, \qquad |a| \leqslant \frac{1}{2}$$

also admits Chebyshev quadrature. Recently, Byrd and Stalla [2] have shown that this result also holds for the weight function

$$(7) \qquad w_3(x) = w_1(x)\frac{1}{2a+1+x}, \qquad a \geqslant 1.$$

There appears to be no other concrete example in the literature of weight functions admitting Chebyshev quadrature.

We now consider a weight function w as a product

$$(8) \qquad w(x) = w_1(x)v(x).$$

Kahaner [12] has shown that for $w \in C(-1,1)$ to admit Chebyshev quadrature, a necessary condition on v is

$$(9) \qquad v(x) \geqslant \tfrac{1}{2}c \quad \text{for all } x \in (-1,1),$$

where

$$c = \int_{-1}^{1} w(x)\,dx \Big/ \int_{-1}^{1} w_1(x)\,dx > 0.$$

Comparing the above weight functions w_1, w_2, and w_3, we see that in these cases v is a rational function continuous on $[-1,1]$.

In this paper we shall prove that the weight functions

$$(10) \qquad w_4(x) = w_1(x)|x|^{-1/2}(1+|x|)^{1/2},$$

$$(11) \qquad w_5(x) = w_1(x)(1+x)^{-1/4}\left(\sqrt{2}+(1+x)^{1/2}\right)^{1/2},$$

$$(12) \qquad w_6(x) = w_1(x)(1-x^2)^{-1/4}\left(1+(1-x^2)^{1/2}\right)^{1/2}$$

also admit Chebyshev quadrature. Our method is quite different from those of Ullman [15] and Byrd and Stalla [2]. With regard to the still open problem of characterizing all weight functions admitting Chebyshev quadrature, it may be of interest that in these three cases the corresponding functions v in (8) have singularities either at an interior point or at one or both of the end points of the interval $[-1,1]$.

After establishing the existence of the Chebyshev quadrature rules for the weight functions (10), (11) and (12), and obtaining their nodes, we shall indicate a simple method for constructing weight functions for which there exist infinitely many Chebyshev quadrature rules. Such examples may also help path the way toward a solution of the above-mentioned problem.

2. Construction of the Chebyshev Rules. We first consider the Chebyshev weight function w_1 given in (5). The corresponding Chebyshev rules Q_n^1 have Gaussian degree of precision $2n-1$, i.e., $R_n^1[f]=0$ for all $f \in \mathscr{P}_{2n-1}$ (see, e.g., Ghizzetti and Ossicini [11, p. 99 ff]). Transforming w_1 and Q_n^1 to the interval $[-1,0]$ as well as to the interval $[0,1]$ and compounding the two resulting weight functions and rules to the interval $[-1,1]$ gives the weight function w_4 together with an equally weighted quadrature rule $\tilde{Q}_{2n}$ having $2n$ nodes and degree of precision $2n-1$. The nodes of the Chebyshev rule Q_n^1 are the zeros of the polynomial T_n, where T_n denotes the Chebyshev polynomial of the first kind of degree n. Hence the nodes of $\tilde{Q}_{2n}$ are the zeros of the polynomial $\tilde{p}_{2n}$ given by

$$\tilde{p}_{2n}(x) = T_n(2x+1)T_n(2x-1). \tag{13}$$

We shall now show that the interpolatory quadrature rule (for definition see, e.g., [1, p. 16]) Q_{2n}^4, whose nodes are the zeros of

$$p_{2n}(x) = \tilde{p}_{2n}(x) - \tfrac{1}{2}, \tag{14}$$

is a Chebyshev quadrature rule with $2n$ nodes for the weight function w_4. If $\tilde{p}_{2n} - \alpha$ ($\alpha \in \mathbf{R}$) has only real zeros, then the interpolatory quadrature formula, whose nodes are the zeros of $\tilde{p}_{2n} - \alpha$, is also equally weighted (see, e.g., [7, p. 103], [9], [4], [5]). So, the proof is completed if it is shown that

(i) $R_{2n}^4[q_{2n}] = 0$, $q_{2n}(x) := x^{2n}$,

(ii) all zeros of p_{2n} are real, pairwise distinct and contained in the open interval $(-1, 1)$.

Because Q_{2n}^4 is an interpolatory quadrature rule, we have $R_{2n}^4[f] = 0$ for all $f \in \mathscr{P}_{2n-1}$ and therefore

$$\begin{aligned} 2^{4n-2}R_{2n}^4[q_{2n}] &= R_{2n}^4[p_{2n}] \\ &= \int_{-1}^1 w_4(x)\left[T_n(2x+1)T_n(2x-1) - \tfrac{1}{2}\right] dx \\ &= -\pi + 2\int_0^1 (x-x^2)^{-1/2} T_n(2x+1)T_n(2x+1)\, dx \\ &= -\pi + 2\int_{-1}^1 w_1(x)T_n(x)T_n(x+2)\, dx \\ &= -\pi + 2\int_{-1}^1 w_1(x)T_n(x)2^{n-1}x^n\, dx \\ &= -\pi + 2\int_{-1}^1 w_1(x)\{T_n(x)\}^2\, dx = 0, \end{aligned}$$

using the known properties of T_n (cf. here and in the following, e.g., Tricomi [14, p. 187 ff] or Paszkowski [13]) as well as the symmetry of w_4 and p_{2n}. This proves (i).

To prove (ii), we need consider only the interval $[0, 1]$ since p_{2n} is symmetric. $T_n(2x-1)$ has in $(0, 1)$ n pairwise distinct real zeros. All $n-1$ relative maxima of $|T_n(2x-1)|$ as well as $T_n(1)$ have the value 1. Since $T_n(2x+1) \geqslant 1$ for all $x \geqslant 0$, all relative maxima of $|\tilde{p}_{2n}|$ as well as $\tilde{p}_{2n}(1)$ have a value not less than 1. Therefore, p_{2n} has all properties required in (ii).

To establish the Chebyshev quadrature rules Q_{2n-1}^4 for the weight function w_4 for all positive n, we consider first the Radau rules for the Chebyshev weight function w_1. They are given by (see, e.g., Ghizzetti and Ossicini [11, p. 101 ff.])

$$Q_n^+[f] = \frac{2\pi}{2n-1}\left\{\frac{1}{2}f(1) + \sum_{\nu=1}^{n-1} f\left(\cos\frac{2\nu}{2n-1}\pi\right)\right\} \tag{15}$$

and

$$Q_n^-[f] = \frac{2\pi}{2n-1}\left\{\frac{1}{2}f(-1) + \sum_{\nu=1}^{n-1} f\left(\cos\frac{2\nu-1}{2n-1}\pi\right)\right\}. \tag{16}$$

We note that the nodes of Q_n^+ are the zeros of $T_n - T_{n-1}$ and that the nodes of Q_n^- are the zeros of $T_n + T_{n-1}$. Both quadrature rules have degree of precision $2n - 2$. Transforming w_1 and Q_n^+ to the interval $[-1, 0]$ and w_1 and Q_n^- to the interval $[0, 1]$ and compounding the two resulting weight functions and rules to the interval $[-1, 1]$ yields the weight function w_4 together with an equally weighted quadrature rule Q_{2n-1}^4 having $2n - 1$ nodes and degree of precision not less than $2n - 2$. Because of the symmetry of w_4 and the symmetry of Q_{2n-1}^4 this quadrature rule has degree of precision $2n - 1$ and is therefore a Chebyshev quadrature rule. We have thus proven the following theorem.

THEOREM 1. *The weight function* $w_4(x) = (1 - x^2)^{-1/2}|x|^{-1/2}(1 + |x|)^{1/2}$ *admits Chebyshev quadrature. The nodes of the corresponding Chebyshev rules* Q_n^4 *are given by the zeros of* p_n^4, *where*

$$p_{2m}^4(x) = T_m(2x - 1)T_m(2x + 1) - \tfrac{1}{2}, \tag{17}$$

$$xp_{2m-1}^4(x) = \left[T_m(2x - 1) + T_{m-1}(2x - 1)\right]\left[T_m(2x + 1) - T_{m-1}(2x + 1)\right]. \tag{18}$$

To establish Chebyshev quadrature for the weight functions w_5 and w_6, the lemma below is helpful (cf. also Gautschi [8, p. 482]).

LEMMA 1. *Let w be a weight function on* $[-1, 1]$ *with* $w(x) = w(-x)$ *for all* $x \in [-1, 1]$ *and let* Q_{2n+1} *be a quadrature rule with respect to w given by*

$$Q_{2n+1}[f] = a_0 f(0) + \sum_{\nu=1}^{n} a_\nu\left[f(x_\nu) + f(-x_\nu)\right] \tag{19}$$

with $0 < x_1 < \cdots < x_n$. *Let* $\tilde{w}(x) := w(\sqrt{x})/\sqrt{x}$ *be a weight function on* $[0, 1]$ *and let* $\tilde{Q}_{n+1}$ *be the quadrature rule with respect to* $\tilde{w}$ *given by*

$$\tilde{Q}_{n+1}[f] = a_0 f(0) + \sum_{\nu=1}^{n} 2a_\nu f(x_\nu^2). \tag{20}$$

Then Q_{2n+1} *has degree of precision* $2m + 1$ *if and only if* $\tilde{Q}_{n+1}$ *has degree of precision* m.

This lemma is well known for $w(x) = 1$ and is used to derive Gauss rules on $[0, 1]$ with respect to the weight function $x^{-1/2}$. (Note that it is possible for a_ν to be zero.)

Application of Lemma 1 to the weight function w_4 and considering the rule $Q_{2n+1} := Q_{2n}^4$ (i.e., $a_0 = 0$) which, because of symmetry, has degree of precision $2n + 1$, gives the weight function

$$\tilde{w}(x) = x^{-3/4}(1 - x)^{-1/2}(1 + \sqrt{x})^{1/2} \tag{21}$$

on the interval $[0, 1]$ and a corresponding equally weighted quadrature rule $\tilde{Q}_n$, whose nodes are the zeros of

$$\tilde{p}_n^4(x) = T_n(2\sqrt{x} - 1)T_n(2\sqrt{x} + 1) - \tfrac{1}{2}. \tag{22}$$

By Lemma 1 the quadrature rule $\tilde{Q}_n$ has degree of precision n. The nodes of Q_{2n}^4 are all pairwise distinct and contained in $(-1, 1)$. So by (20), the n nodes of $\tilde{Q}_n$ are also pairwise distinct and contained in $(0, 1)$. Transforming to the interval $[-1, 1]$ yields the following theorem.

THEOREM 2. *The weight function*

$$w_5(x) = (1 - x^2)^{-1/2}(1 + x)^{-1/4}\left(\sqrt{2} + (1 + x)^{1/2}\right)^{1/2}$$

admits Chebyshev quadrature. The nodes for the corresponding Chebyshev rules Q_n^5 *are given by the zeros of* p_n^5, *where*

$$p_n^5(x) = T_n(\sqrt{2x + 2} - 1)T_n(\sqrt{2x + 2} + 1) - \tfrac{1}{2}. \tag{23}$$

Using (21), let $\bar{w}$ be defined by

$$\bar{w}(x) = \tilde{w}(1 - x) = x^{-1/2}(1 - x)^{-3/4}(1 + \sqrt{1 - x})^{1/2}. \tag{24}$$

Since $\tilde{w}$ is a weight function on $[0, 1]$ admitting Chebyshev quadrature, so is $\bar{w}$. By (22), the nodes of the corresponding Chebyshev rules $\bar{Q}_n$ are the zeros of

$$\bar{p}_n(x) = T_n(2\sqrt{1 - x} - 1)T_n(2\sqrt{1 - x} + 1) - \tfrac{1}{2}. \tag{25}$$

Applying Lemma 1 ($a_0 = 0$) to $\bar{w}$ and $\bar{Q}_n$ gives the weight function w_6 on $[-1, 1]$ and the corresponding Chebyshev rule Q_{2n}^6.

To establish the Chebyshev quadrature rules Q_{2n-1}^6, we consider again the weight function w_1 and the corresponding Radau rules Q_n^+ and Q_n^- in (15) and (16). Transforming w_1 and Q_n^- to the interval $[-1, 0]$ and w_1 and Q_n^+ to the interval $[0, 1]$ and compounding yields the weight function w_4 in $[-1, 1]$ together with a quadrature rule Q_{2n}^*. Owing to symmetry, the rule Q_{2n}^* has degree of precision $2n - 1$. The $2n - 2$ nodes in $(-1, 1)$ are equally weighted, for the nodes -1 and 1 the weights are half as large as the weight of the other nodes. Applying Lemma 1 again gives on $[0, 1]$ the weight function $\tilde{w}$ in (21) and a quadrature rule $\tilde{Q}_n^*$ having degree of precision $n - 1$. $\tilde{Q}_n^*$ has in $(0, 1)$ $n - 1$ equally weighted nodes; for the node 1 the weight is half as large as the weight of the other nodes. With the help of the transformation $y = 1 - x$ we obtain on $[0, 1]$ the weight function $\bar{w}$ in (24) together with a corresponding quadrature rule Q_n^*. Applying Lemma 1 again now yields the weight function w_6 and the Chebyshev quadrature rule Q_{2n-1}^6.

THEOREM 3. *The weight function* $w_6(x) = (1 - x^2)^{-3/4}(1 + (1 - x^2)^{1/2})^{1/2}$ *admits Chebyshev quadrature. The nodes of the corresponding Chebyshev rules* Q_n^6 *are given by the zeros of* p_n^6, *where*

$$p_{2m}^6(x) = T_m(2\sqrt{1 - x^2} - 1)T_m(2\sqrt{1 - x^2} + 1) - \tfrac{1}{2}, \tag{26}$$

$$\begin{aligned} xp_{2m-1}^6(x) = &\left[T_m(2\sqrt{1 - x^2} - 1) - T_{m-1}(2\sqrt{1 - x^2} - 1)\right] \\ &\cdot\left[T_m(2\sqrt{1 - x^2} + 1) + T_{m-1}(2\sqrt{1 - x^2} + 1)\right]. \end{aligned} \tag{27}$$

In connection with the open problem of characterizing all weight functions admitting Chebyshev quadrature, we mention that the Jacobi weight function $(1 - x^2)^{-3/4}$ does not admit Chebyshev quadrature [6].

Remarks. (a) Note that the polynomial p_{2n}^4 is symmetric on the interval $[-1, 1]$ and has even degree $2n$. Hence $\tilde{p}_n^4$, p_n^5, $\bar{p}_n$, and p_{2n}^6 are also polynomials because of the identities $\tilde{p}_n^4(x) = p_{2n}^4(\sqrt{x})$, $p_n^5(x) = p_{2n}^4(\sqrt{2x + 2})$, $\bar{p}_n = p_{2n}^4(\sqrt{1 - x})$, and $p_{2n}^6(x) = p_{2n}^4(\sqrt{1 - x^2})$. Applying the same reasoning, p_{2n-1}^6 can also be shown to be a polynomial by virtue of the identity $xp_{2n-1}^6(x) = q_{2n}(\sqrt{1 - x^2})$, where q_{2n} is defined by

$$q_{2n}(x) = \left[T_n(2x - 1) - T_{n-1}(2x - 1)\right]\left[T_n(2x + 1) + T_{n-1}(2x + 1)\right],$$

since again, q_{2n} is a symmetric polynomial of even degree $2n$.

(b) Since $T_n(x) = \cos(n \arccos x)$, the zeros of p^4_{2m-1} and p^6_{2m-1} can be obtained explicitly. Using the identity (see, e.g., [13, p. 22])

$$T_n(2x-1) + T_{n-1}(2x-1) = T_{2n}(\sqrt{x}) + T_{2n-2}(\sqrt{x}) = 2\sqrt{x}\, T_{2n-1}(\sqrt{x}),$$

we have that the zeros of the symmetric polynomial p^4_{2m-1} agree with those of the function t_{2m-1}, where t_{2m-1} is defined by

$$t_{2m-1}(x) = T_{2m-1}\left(\sqrt{|x|}\right);$$

and by similar argument, that the zeros of p^6_{2m-1} agree with those of $\bar{t}_{2m-1}$, where $\bar{t}_{2m-1}$ is defined by

$$\bar{t}_{2m-1}(x) = T_{2m-1}\left(\sqrt{1-(1-x^2)^{1/2}}\right), \qquad |x| \leqslant 1.$$

(c) Let W be defined by $W(x) = w(-x)$. If w admits Chebyshev quadrature, then so does W. This follows by transforming w and Q_n by a reflection at the origin on the interval $[-1,1]$. Applying this argument to w_5 shows that $\overline{w}_5$ also admits Chebyshev quadrature, where $\overline{w}_5$ is defined by

$$\overline{w}_5(x) = w_1(x)(1-x)^{-1/4}\left(\sqrt{2} + (1-x)^{1/2}\right)^{1/2}.$$

3. Construction of Weight Functions Having Infinitely Many Chebyshev Quadrature Rules. Weight functions admitting Chebyshev quadrature are rare (Gautschi [7, p. 109]). Therefore, one may seek weight functions having infinitely many Chebyshev quadrature rules. Apart from the weight functions w_1, w_2, and w_3, the author has found in the literature only three other weight functions having this property. They are given by Geronimus [10] as follows:

$$w_A(x) = w_1(x)\frac{1-a+2ax^2}{(1-a)^2+4ax^2}, \qquad |a| < \frac{1}{2}, \tag{28}$$

$$w_B(x) = \begin{cases} 0 & \text{for all } x \in (-\alpha,\alpha),\ 0<\alpha<1, \\ w_1(x)(x^2-\alpha^2)^{-1/2}|x|\dfrac{(1-\alpha^2)(1-a)+2a(x^2-\alpha^2)}{(1-\alpha^2)(1-a)+4a(x^2-\alpha^2)}, & \\ & |a| < \dfrac{1}{2}, \end{cases} \tag{29}$$

$$\begin{gathered} w_C(x) = w_1(x)\frac{1+a^2-2ax}{1+b^2-2bx}, \\ a = \frac{2+b^2-\sqrt{(1-b^2)(4-b^2)}}{3b}, \qquad |b|<1,\ |a|<1. \end{gathered} \tag{30}$$

For each of these three weight functions, Chebyshev quadrature rules Q_n exist for every even n. In the case of w_B with $a = 0$ see also Gautschi [8, p. 483], where the Gaussian degree $2n-1$ for even n has been proved.

We now indicate a simple method for constructing other weight functions having infinitely many Chebyshev quadrature rules. Given a weight function w_a on $[-1,1]$ of the form

$$w_a(x) = w_1(x)v_a(x) \tag{31}$$

having a Chebyshev quadrature rule Q_n^a, we transform w_a and Q_n^a to the interval [0, 1] and apply Lemma 1. We obtain on [−1, 1] the weight function

$$w_b(x) = w_1(x)v_a(2x^2 - 1) = w_1(x)v_a(T_2(x)) \tag{32}$$

and a corresponding Chebyshev quadrature rule Q_{2n}^b. Repeating this procedure and noting that

$$T_n(T_m) = T_{nm}, \tag{33}$$

we obtain the following theorem.

THEOREM 4. *Let $w_a(x) = w_1(x)v_a(x)$ be a weight function having a Chebyshev quadrature rule with n nodes and let $k \in \mathbf{N}$. Then the weight function*

$$w_b(x) = w_1(x)v_a(T_{2^k}(x)) \tag{34}$$

has a Chebyshev quadrature rule with $2^k n$ nodes.

As a first example, we apply Theorem 4 to the weight function w_2 of Ullman given in (6). In the special case $k = 1$ we arrive at the weight function w_A in (28) and the corresponding result of Geronimus [10].

Applying Theorem 4 to the weight function w_3 of Byrd and Stalla [2] for $k = 1$, we obtain that the weight function

$$w_D(x) = w_1(x)\frac{1}{a + x^2}, \qquad a \geqslant 1, \tag{35}$$

has a Chebyshev quadrature rule Q_n^D for every even $n \in \mathbf{N}$.

In the case of weight functions w_4 and w_6 we arrive at the weight functions

$$w_E(x) = w_1(x)|T_{2^k}(x)|^{-1/2}(1 + |T_{2^k}(x)|)^{1/2}, \qquad k \in \mathbf{N}, \tag{36}$$

$$w_F(x) = (1 - x^2)^{-3/4}|U_{2^k-1}(x)|^{-1/2}\{1 + (1 - x^2)^{1/2}|U_{2^k-1}(x)|\}^{1/2}, \qquad k \in \mathbf{N}, \tag{37}$$

having for every $n \in \mathbf{N}$ a Chebyshev quadrature rule with $2^k n$ nodes. (U_m denotes the Chebyshev polynomial of the second kind of degree m.) With the help of w_E resp. w_F we see that for every $k \in \mathbf{N}$ there exists a weight function admitting infinitely many Chebyshev quadrature rules, which has 2^k resp. $2^k - 1$ pairwise distinct singularities in $(-1, 1)$.

Finally, we mention two generalizations of the above principle for the construction of weight functions admitting Chebyshev quadrature. Instead of transforming w_a and Q_n^a in (31) to the interval [0, 1] we transform both to the interval $[\alpha, 1]$, $0 \leqslant \alpha < 1$, and then apply Lemma 1. For example, in the case of the weight function w_2, we arrive at the weight function w_B in (29) and the corresponding result of Geronimus [10]. A further variation is given by the additional transformation $y = \alpha + 1 - x$ before applying Lemma 1. If w_s is nonsymmetric on $[-1, 1]$, this also leads to new weight functions admitting Chebyshev quadrature.

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